

Contraction algebra and invariants of singularities

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Abstract

In [7], Donovan and Wemyss introduced the contraction algebra of flopping curves in 3-folds. When the flopping curve is smooth and irreducible, we prove that the contraction algebra together with its A_∞ -structure recovers various invariants associated to the underlying singularity and its small resolution, including the derived category of singularities and the genus zero Gopakumar-Vafa invariants.

1 Introduction

1.1 Motivation

In their recent works [7], [8], Donovan and Wemyss constructed certain algebras called *contraction algebras* associated with birational morphisms $f : X \rightarrow Y$ with at most one-dimensional fibers. The contraction algebras prorepresent the functors of noncommutative deformations of the reduced fiber of f . When f is a 3-fold flopping contraction, they described the Bridgeland-Chen's flop-flop autoequivalence of the derived category of coherent sheaves on X in terms of the twisted functor associated with the contraction algebra. This is a remarkable application of non-commutative deformation theory first developed by Laudal to the commutative algebraic geometry, and provides a deep understanding of floppable $(1, -3)$ -curves, mysterious curves in birational geometry. Indeed they conjectured that their contraction algebra recovers the formal neighborhood at the flopping curve, so that the contraction algebra contains enough information for the local geometry of the flopping curve. The above Donovan-Wemyss's conjecture is one of the motivations in this article.

The Donovan-Wemyss's contraction algebra is also interesting in enumerative geometry. As an analogy of Reid's width for a $(0, -2)$ -curve, they introduced

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the new invariant for a $(1 - 3)$ -curve called the non-commutative width as the dimension of the contraction algebra. In [21], the second author showed that their non-commutative widths are described by weighted sums of Katz's genus zero Gopakumar-Vafa (GV) invariants, which virtually count rational curves on Calabi-Yau 3-folds. The formula proved in [21] (cf. Theorem 4.5) is quite simple, but the proof of [21] was not enough intrinsic to explain its meaning. Also it was not shown whether the contraction algebra recovers the GV invariants of all degrees or not. This article is also motivated by the above questions on the relationship between contraction algebras and GV invariants.

1.2 Results and the organization of the paper

After reviewing some basic terminology in Section 2, we study an alternative description of contraction algebras in Section 3. We will show that the contraction algebras are described in terms of (derived) endomorphism algebras in Orlov's derived category of singularities $D_{sg}(Y)$ of Y (Corollary 3.3). This result will be used to provide some additional structures on contraction algebras. We see that our alternative description of the contraction algebra turns it into a $\mathbb{Z}/2$ -graded A_∞ -algebra with a Frobenius structure, which we call the ∞ -contraction algebra. It reduces to Donovan-Wemyss's contraction algebra by forgetting the higher A_∞ -structures. We prove that the ∞ -contraction algebra recovers the derived category of singularities $D_{sg}(Y)$ and the triangulated subcategory $\mathcal{C}^b \subset D^b(\text{coh} X)$ of objects F with $\mathbf{R}f_* F = 0$ (Theorem 3.7).

In Section 4, we use the above alternative description of the contraction algebras to give their deformations to semisimple algebras with the numbers of blocks coincide with the sums of the GV invariants (Theorem 4.2). We are still not able to prove that our deformations of the contraction algebras are flat, but assuming this would give a different and more intrinsic proof of the dimension formula of the contraction algebra given in [21]. We will also address the question regarding the reconstruction problem of GV invariants from the contraction algebras. Using the work of the second author on wall crossing formula of parabolic stable pairs [19], we show that the generating series of non-commutative Hilbert schemes associated with contraction algebras admit a product expansion whose power is given by GV invariants (Theorem 4.4). This result also leads to an alternative proof of the dimension formula in [21].

In Section 5, we study Donovan-Wemyss's conjecture (Conjecture 5.3), which says that the contraction algebra determines the formal neighborhood of the flopping curve. We will propose an A_∞ -enhancement (Conjecture 5.3) of the above conjecture, using the A_∞ -structure of the contraction algebra. We prove it for the special case when the singularity of Y is weighted homogeneous (Theorem 5.5). Finally in Section 6, we study several examples of contraction algebras, including flopping contractions of type cA_1 , cD_4 .

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2 Preliminary

2.1 Contraction algebra

Let X be a smooth quasi-projective complex 3-fold. A flopping contraction is a birational morphism

$$f : X \rightarrow Y \quad (2.1)$$

which is an isomorphism in codimension one, Y has only Gorenstein singularities and the relative Picard number of f equals to one. We denote by C the exceptional locus of f , which is a tree of smooth rational curves. Below, we assume that Y is an affine variety and the exceptional locus C is isomorphic to \mathbb{P}^1 . Let \mathcal{L} be a relative ample line bundle on X . Take a minimal generating set of $\text{Ext}^1(\mathcal{O}_X, \mathcal{L}^{-1})$ as $\Gamma(\mathcal{O}_X)$ -module consisting of r elements. We define the vector bundle \mathcal{N} to be the extension:

$$0 \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{N} \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow 0 \quad (2.2)$$

We set $\mathcal{U} := \mathcal{O}_X \oplus \mathcal{N}$, $N := \mathbf{R}f_*\mathcal{N} = f_*\mathcal{N}$ and

$$A := \text{End}_X(\mathcal{U}) \cong \text{End}_Y(\mathcal{O}_Y \oplus N).$$

By Van den Bergh ([22, Section 3.2.8]), \mathcal{U} is a tilting object. The functor $F := \mathbf{R}\text{Hom}_X(\mathcal{U}, -)$ defines an equivalence of categories $\text{D}^b(\text{coh}(X)) \cong \text{D}^b(\text{mod-}A)$. Obviously, $F(\mathcal{O}_X)$ and $F(\mathcal{N})$ are projective A -modules.

Let $p \in Y$ be the image of C under f , and we set $R = \hat{\mathcal{O}}_{Y,p}$ the formal completion of \mathcal{O}_Y at the singular point p . We take the completion of 2.1

$$\hat{f} : \hat{X} := X \times_Y \text{Spec } R \rightarrow \hat{Y} := \text{Spec } R. \quad (2.3)$$

Let \mathcal{L} be a relative ample line bundle on \hat{X} such that $\deg(\mathcal{L}|_C) = 1$. In this case,

$$A := \text{End}_{\hat{X}}(\mathcal{U}) \cong \text{End}_R(R \oplus N).$$

The following proposition is due to Van den Bergh [22].

Proposition 2.1. ([22, Proposition 3.5.7]) *Let $f : \hat{X} \rightarrow \hat{Y}$ be the flopping contraction as before. Denote C_l for the scheme theoretical fiber of $p \in Y$. Then $F(\mathcal{O}_{C_l})$ and $F(\mathcal{O}_C(-1)[1])$ are simple A -modules.*

Definition 2.2. ([7, Definition 2.8]) The contraction algebra A_{con} is defined to be A/I_{con} , where I_{con} is the two sided ideal of A consisting of morphisms $R \oplus N \rightarrow R \oplus N$ factoring through a summand of finite sums of R .

The contraction algebra A_{con} is an augmented algebra. The augmentation morphism is defined by the algebra morphism $A \rightarrow \text{End}_A(S) \cong \mathbb{C}$, which factors through A_{con} . Here $S = F(\mathcal{O}_C(-1)[1])$ is the simple A -module in Proposition 2.1.

Remark 2.3. In [7], Donovan and Wemyss gave an alternative definition of the contraction algebra using noncommutative deformation functor (see Definition 2.11 [7]). It is Morita equivalent with the contraction algebra in Definition 2.2 but may not be isomorphic. In this paper, we will use only Definition 2.2 (and its variation in Theorem 2.5). The Morita equivalence of the contraction algebras in these two definitions can be explained by Theorem 4.2.

2.2 Categories associated to the singularity and its resolution

Let Y be a quasi-projective scheme. Denote $D^b(\text{coh}(Y))$ for the bounded derived category of coherent sheaves on Y and $\mathfrak{P}\text{erf}(Y)$ for the full subcategory consisting of perfect complexes on Y . We define a triangulated category $D_{sg}(Y)$ as the quotient of $D^b(\text{coh}(Y))$ by $\mathfrak{P}\text{erf}(Y)$ (cf. [17, Definition 1.8]). Consider the case when Y is a hypersurface in a smooth affine variety $\text{Spec } B$ defined by $W = 0$ for a function $W \in B$. Orlov proved that the derived category of singularities $D_{sg}(Y)$ is equivalent, as triangulated categories, with the homotopy category of the category of matrix factorizations $\text{MF}(W)$ (cf. [17, Theorem 3.9]). The objects of $\text{MF}(W)$ are the ordered pairs:

$$\bar{E} = (E, \delta_E) := E_1 \begin{array}{c} \xrightarrow{\delta_1} \\ \xleftarrow{\delta_0} \end{array} E_0$$

where E_0 and E_1 are finitely generated projective B -modules and the compositions $\delta_0\delta_1$ and $\delta_1\delta_0$ are the multiplications by the element $W \in B$. The precise definition of the category of matrix factorizations $\text{MF}(W)$ can be found in [17, Section 3].

Using the category of singularities, one can give an alternative definition for the contraction algebra A_{con} . First we recall a lemma of Orlov about category of singularities for Gorenstein schemes.

Proposition 2.4. ([17, Proposition 1.21]) *Let Y be a Gorenstein scheme of finite Krull dimension. Let \mathcal{F} and \mathcal{G} be coherent sheaves on Y such that $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_Y) = 0$ for all $i > 0$. Fix N such that $\text{Ext}^i(\mathcal{P}, \mathcal{G}) = 0$ for $i > N$ and for any locally free sheaf \mathcal{P} . Then*

$$\text{Hom}_{D_{sg}}(\mathcal{F}, \mathcal{G}[N]) \cong \text{Ext}^N(\mathcal{F}, \mathcal{G})/\mathcal{R}$$

where \mathcal{R} is the subspace of elements factoring through locally free.

The condition $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_Y) = 0$ is satisfied when \mathcal{F} is maximal Cohen-Macaulay. If we assume that Y is affine then the integer N can be chosen to be 0.

Theorem 2.5. *Let $f : X \rightarrow Y$ be a flopping contraction defined as above such that $Y = \text{Spec } R$ is affine. There is a natural isomorphism*

$$A_{con} \cong \text{Hom}_{D_{sg}(Y)}^0(N, N).$$

Proof. The quotient functor $D^b(\text{coh}(Y)) \rightarrow D_{sg}(Y)$ induces a map from A to $\text{Hom}_{D_{sg}}^0(R \oplus N, R \oplus N) \cong \text{Hom}_{D_{sg}}^0(N, N)$. By local duality

$$\mathbf{R}f_* \mathbf{R}\mathcal{H}om_X(\mathcal{N}, f^! \mathcal{O}_Y) \simeq \mathbf{R}\mathcal{H}om(f_* \mathcal{N}, \mathcal{O}_Y),$$

$\text{Ext}_R^{>0}(N, R) = 0$. Proposition 2.4 implies that for $i \geq 0$

$$\text{Hom}_{D_{sg}}(N, N[i]) \cong \text{Ext}_R^i(N, N)/\mathcal{R}$$

where \mathcal{R} are morphisms from N to N that factors through a projective R module. Take $i = 0$ and the theorem is proved. \square

Below, we assume that Y is an affine hypersurface with one isolated singular point. By taking the completion at the singular point, we may further assume that $B = \mathbb{C}[[x_1, \dots, x_n]]$ and W has an isolated critical point at the origin. We define the *Milnor algebra* of Y to be $B_W := \mathbb{C}[[x_1, \dots, x_n]]/J_W$ with $J_W := (\partial_1 W, \dots, \partial_n W)$. In this case, it is well known that the Hochschild homology of $\text{MF}(W)$ is isomorphic, as $\mathbb{Z}/2$ -graded vector spaces, with B_W (for example, see [18, Section 2.4]). This is in fact a consequence of the following theorem due to Dyckerhoff:

Theorem 2.6. ([6, Theorem 4.1]) *Let (B, W) be defined as before and consider the residue field \mathbb{C} as a $B/(W)$ -module. Denote the corresponding matrix factorization by k^{st} . Then k^{st} is a compact generator of the triangulated category of $\text{MF}(W)$.*

If $f : X \rightarrow Y$ is a projective morphism of quasi-projective varieties such that $\mathbf{R}f_* \mathcal{O}_X = \mathcal{O}_Y$, then the functor

$$\mathbf{L}f^* : D(Y) \rightarrow D(X)$$

embeds $D(Y)$ as a right admissible triangulated subcategory of $D(X)$. In particular, this holds for the three dimensional flopping contraction. Define a subcategory \mathcal{C} of $D(X)$ by

$$\mathcal{C} = \{E \in D(X) : \mathbf{R}f_*(E) = 0\}.$$

This is the left orthogonal of $\mathbf{L}f^* D(Y)$. The objects in \mathcal{C} are support on the exceptional locus of f .

Denote \mathcal{C}^b for the subcategory of \mathcal{C} whose objects E have bounded coherent cohomologies. In the case when $f : X \rightarrow Y$ is a three dimensional flopping contraction with the exceptional curve denoted by C , we have:

Proposition 2.7. *Any object $E \in \mathcal{C}^b$ is a consequent extensions of shifts of $\mathcal{O}_C(-1)$.*

Proof. This is an immediate consequence of Proposition 2.1. \square

The above proposition shows that \mathcal{C}^b is generated by the compact object $\mathcal{O}_C(-1)$. Let \mathcal{D} be a dg-category, \mathbb{Z} -graded or $\mathbb{Z}/2$ -graded. If G is a compact generator of \mathcal{D} then \mathcal{D} is quasi-equivalent with the category of dg-modules over the \mathbb{Z} -graded or $\mathbb{Z}/2$ -graded dg-algebra $\mathrm{Hom}_{\mathcal{D}}^{\bullet}(G, G)$. This is a consequence of the derived Morita theory of dg-categories. The $\mathbb{Z}/2$ -graded variation of it can be found in [6, Chapter 5]. In the upcoming section, we will link the categories $\mathrm{MF}(W)$ and \mathcal{C}^b to the contraction algebra by choosing appropriate compact generators.

3 Structures on the contraction algebra

Assume that $f : X \rightarrow Y$ is a three dimensional flopping contraction for $Y = \mathrm{Spec} R$ and $R = \mathbb{C}[[x, y, z, w]]/(W)$. A coherent sheaf E on Y can be identified with a finitely generated R -module M . It represents an object in the category $\mathrm{D}_{sg}(Y)$, which is equivalent with $\mathrm{MF}(W)$. We denote the corresponding matrix factorization of M by M^{st} , called the *stabilization* of M .

From the theorem of Dyckerhoff, we know that k^{st} is a compact generator of $\mathrm{MF}(W)$ for any isolated hypersurface singularities. Though k^{st} contains all the information of the hypersurface singularity, it is usually hard to compute. For the singularity underlying a three dimensional flopping contraction, a different generator exists by the work of Iyama and Wemyss [11].

Proposition 3.1. *Let $\mathcal{U} = \mathcal{N} \oplus \mathcal{O}_X$ be the tilting bundle on X and N be the R -module defined as $f_*\mathcal{N}$. Its stabilization N^{st} is a compact generator of $\mathrm{MF}(W)$.*

Proof. Because $R \oplus N$ is a tilting object, proposition 5.10 of [11] implies that N is a generator. By [6, Section 4], N^{st} is a compact object. \square

For a general R -module M , the derived pull back $\mathbf{L}f^*M$ is unbounded. However, for those modules that lie in the perverse heart in the sense of [22, Section 3]. We have the following vanishing properties for the derived pullbacks proved by Bodzenta and Bondal.

Theorem 3.2. ([3, Lemma 3.5, 3.6]) *Suppose $f : X \rightarrow Y$ is a projective birational morphism of relative dimension one of quasi-projective Gorenstein varieties of dimension $n \geq 3$. The exceptional locus of f is of codimension greater than one in X . Variety Y has canonical hypersurface singularities of multiplicity two. Then $L^{-1}f^*f_*\mathcal{N} = 0$ for any object \mathcal{N} in the heart of ${}^{-1}\mathrm{Per}(X/Y)$.*

Notice that the assumption of the above theorem is satisfied for the three dimensional flopping contraction.

Corollary 3.3. *Let \mathcal{N} and N be defined in the previous section. Then we have $\text{Ext}_{D_{sg}}^1(N, N) = 0$ and*

$$\text{Hom}_{D_{sg}}^\bullet(N, N) = \text{Hom}_{D_{sg}}^0(N, N) = A_{con}. \quad (3.1)$$

Proof. The result of Van den Bergh shows that the vanishing theorem in Theorem 3.2 is applicable to \mathcal{N} . By proposition 2.4, it suffices to show that $\text{Ext}_R^1(N, N)$ vanishes. By adjunction, $\text{Ext}_R^1(N, N) \cong \text{Ext}_X^1(\mathbf{L}f^*f_*\mathcal{N}, \mathcal{N})$. The vanishing $\text{Ext}_{D_{sg}}^1(N, N) = 0$ follows from Theorem 3.2 and the local to global spectral sequence. The identities (3.1) now follow from Theorem 2.5 and the above vanishing. \square

Applying the derived Morita theory of $\mathbb{Z}/2$ -graded dg-category (c.f. [6, Chapter 5]), proposition 3.1 implies that $\text{MF}(W)$ is equivalent to the category of dg-modules over the $\mathbb{Z}/2$ -graded dg-algebra $\text{Hom}_{\text{MF}(W)}^\bullet(N^{st}, N^{st}) \cong \text{Hom}_{D_{sg}}^\bullet(N, N)$. We denote A_{con}^∞ for the minimal model of $\text{Hom}_{D_{sg}}^\bullet(N, N)$ and call it the ∞ -contraction algebra. The contraction algebra A_{con} is the algebra underlying A_{con}^∞ forgetting the higher Massey products. In particular, A_{con}^∞ and A_{con} have the same underlying vector space. The A_∞ -structure on A_{con}^∞ is always nontrivial unless the singularity is non-degenerated (which means the Hessian is non-degenerated). We will study this in Section 5 by one example. Moreover, the A_∞ -structure on A_{con}^∞ is conjectured to reconstruct the local ring of the singularity (see conjecture 5.3).

In the rest of this section, we construct a Frobenius structure on A_{con} . Let W be a function in $B := \mathbb{C}[[x_1, \dots, x_n]]$ with isolated critical point at the origin. Recall from [18, Section 2.4] that

$$\text{HH}_\bullet(\text{MF}(W)) \cong B_W \cdot d\mathbf{x}[n],$$

where $d\mathbf{x} = dx_1 \wedge \dots \wedge dx_n$.

Given a matrix factorization $\bar{E} = (E, \delta_E)$, its chern character $ch(\bar{E}) \in \text{HH}_0(\text{MF}(W))$ is defined to be

$$ch(\bar{E}) = \text{str}(\partial_n \cdot \dots \cdot \partial_1) \cdot d\mathbf{x} \text{ mod } J_W \cdot d\mathbf{x}$$

where str is the supertrace of the matrix with entries in B . There is a canonical bilinear form on $\text{HH}_\bullet(\text{MF}(W))$ with the form:

$$\langle f \otimes d\mathbf{x}, g \otimes d\mathbf{x} \rangle = (-1)^{\frac{n(n-1)}{2}} \text{tr}(f \cdot g),$$

where tr is the trace on the Milnor algebra defined by the generalized residue:

$$\text{tr}(f) = \text{Res} \left[\frac{f(x) \cdot dx_1 \wedge \dots \wedge dx_n}{\partial_1 W, \dots, \partial_n W} \right].$$

The Hirzebruch-Riemann-Roch formula for matrix factorization, proved by Polishchuk and Vaintrob ([18]) says:

$$\chi(\bar{E}, \bar{F}) = \langle ch(\bar{E}), ch(\bar{F}) \rangle. \quad (3.2)$$

More generally, we can define a canonical “boundary-bulk” map

$$\tau^{\bar{E}} : \text{Hom}^\bullet(\bar{E}, \bar{E}) \rightarrow \text{HH}_\bullet(\text{MF}(W))$$

by

$$\tau^{\bar{E}}(\alpha) = \text{str}(\partial_n \delta_E \cdots \partial_1 \delta_E \circ \alpha) \cdot d\mathbf{x} \bmod J_W \cdot d\mathbf{x}.$$

It is clear that the chern character $ch(\bar{E})$ is a special case of $\tau^{\bar{E}}$ when α equals to the identity map of \bar{E} .

By theorem 2.6, $\text{MF}(W)$ is a saturated dg-category. A fundamental theorem of Auslander [2] states the Serre functor \mathbf{S} on $\text{MF}(W)$ is $(-)[n-2]$. Then there is a perfect pairing:

$$\sigma : \text{Hom}^i(\mathbf{S}(\bar{E}), \bar{F}) \otimes_{\mathbb{C}} \text{Hom}^i(\bar{F}, \bar{E}) \rightarrow \mathbb{C}$$

for all i . Because $\text{MF}(W)$ is 2-periodic, the Serre functor is the identity functor if $n = 4$. Apply to the case $\bar{E} = \bar{F} = N^{st}$, we have a perfect pairing:

$$\sigma : A_{con} \otimes_{\mathbb{C}} A_{con} \rightarrow \mathbb{C}$$

In [16], Murfet constructs σ explicitly in term of the boundary bulk map. Apply Murfet’s construction to our situation gives the following. Given α and β in A_{con} , the pairing is

$$\sigma(\alpha, \beta) = \text{Res}(\tau^{N^{st}}(\alpha \circ \beta)).$$

The following proposition follows immediately.

Proposition 3.4. *The contraction algebra A_{con} together with the bilinear form σ forms a Frobenius algebra.*

The Milnor algebra B_W is a commutative Frobenius algebra. The contraction can be viewed as a noncommutative analogue of B_W . We will show later that compare with B_W , the Frobenius algebra A_{con} behaves better under deformations. Like B_W , the contraction algebra A_{con} is also augmented. Given $\alpha \in A_{con}$, we call $\text{Res}(\tau^{N^{st}}(\alpha))$ the *trace* of α . The following remark shows that there exists a canonical element (up to scalar) in A_{con} with nonzero trace.

Remark 3.5. Denote the Jacobson ideal of A_{con} by $J(A_{con})$. It is nothing but the kernel of the augmentation map. Denote $\text{soc}(A_{con})$ for the left ideal of A (treated as a submodule of ${}_A A$) that is annihilated by $J(A_{con})$. By corollary 5.7 [7], A_{con} is a self-injective algebra (therefore Auslander-Gorenstein). Denote the simple right A_{con} -module $F(\mathcal{O}_C(-1)[1])$ by S . Apply $\text{Hom}(-, A_{con})$ to the short exact sequence of right A_{con} -modules:

$$0 \longrightarrow J(A_{con}) \longrightarrow A_{con} \longrightarrow S \longrightarrow 0$$

By the self-injective property, $\text{soc}(A_{con})$ is isomorphic to $\text{Hom}_{A_{con}}(S, A_{con})$ as left A_{con} -modules. Therefore, $\text{soc}(A_{con})$ is one dimensional over \mathbb{C} . Choose a generator α of $\text{soc}(A_{con})$. Because $\text{soc}(A_{con})$ is contained in all the nonzero ideals of A_{con} , the Frobenius property implies that $\text{Res}(\tau^{N^{st}}(\alpha))$ is non zero. Moreover, $\tau^{N^{st}}(\text{soc}(A_{con})) = \text{soc}(B_W)$.

Proposition 3.6. *The boundary bulk map $\tau^{N^{st}}$ factors through $A_{con}/[A_{con}, A_{con}]$.*

Proof. By Lemma 1.1.3 and Lemma 1.2.4 of [18], $\tau^{N^{st}}$ can be interpreted as the trace of a natural transform from the functor $\mathbf{R}\mathrm{Hom}_{\mathrm{MF}(W)}(N^{st}, -) \otimes N^{st}$ to the identity functor. Therefore, it is zero on $[A_{con}, A_{con}]$. \square

To finish this section, we prove that the category \mathcal{C}^b defined in the previous section can be reconstructed from the contraction algebra. Let S be the simple A -module defined in remark 3.5. Because $\mathrm{Hom}_X(\mathcal{O}_C(-1), \mathcal{O}_X) = 0$, S is an A_{con} -module. The surjection $A \rightarrow A_{con}$ defines a functor $j_* : \mathrm{mod}\text{-}A_{con} \rightarrow \mathrm{mod}\text{-}A$. Under this notation, j_*S is simply S treated as an A -module. Because $\mathcal{O}_C(-1)$ is a compact generator of \mathcal{C}^b , one can reconstruct \mathcal{C}^b from the A_∞ -algebra $\mathrm{Ext}_X^\bullet(\mathcal{O}_C(-1), \mathcal{O}_C(-1))$, which is isomorphic to $\mathrm{Ext}_A^\bullet(j_*S, j_*S)$ under the equivalence $F[1]$. The reconstruction theorem of \mathcal{C}^b from A_{con} can be proved by relating the deformation theory of S and that of j_*S .

Theorem 3.7. *Suppose that $X \rightarrow Y$ and $X' \rightarrow Y'$ are three dimensional flopping contractions in smooth quasi-projective 3-folds, to points p and p' respectively. Assume that Y and Y' are spectrums of complete local rings. To these, associate the contraction algebras A_{con} and A'_{con} . Denote the category \mathcal{C}^b associated to X and X' by $\mathcal{C}^b(X)$ and $\mathcal{C}^b(X')$. They are equivalent if $A_{con} \cong A'_{con}$.*

Proof. As \mathcal{C}^b is equivalent to the triangulated category of the category of modules over the A_∞ -algebra $\mathrm{Ext}_A^\bullet(j_*S, j_*S)$, it suffices to show this algebra (together with its A_∞ -structure) can be reconstructed from A_{con} .

By the work of Keller [13], the functor $j_* : \mathrm{mod}\text{-}A_{con} \rightarrow \mathrm{mod}\text{-}A$ admits a dg-lifting. Therefore, it induces a morphism of A_∞ -algebras from $\mathrm{Ext}_{A_{con}}^\bullet(S, S)$ to $\mathrm{Ext}_A^\bullet(j_*S, j_*S)$. By adjunction,

$$\mathrm{Ext}_A^\bullet(j_*S, j_*S) \simeq \mathrm{Ext}_{A_{con}}^\bullet(\mathbf{L}j^*j_*S, S).$$

There is a spectral sequence with E_2 page

$$\mathrm{Ext}_{A_{con}}^p(\mathbf{L}^{-q}j^*j_*S, S)$$

converge to $\mathrm{Ext}_A^{p+q}(j_*S, j_*S)$. By proposition 5.6 of [7], A_{con} admits a projective resolution of A -modules of following

$$0 \longrightarrow P \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow P \longrightarrow A_{con} \longrightarrow 0$$

where $P = F(\mathcal{N})$, Q_0 and Q_1 are direct summand of sum of $F(\mathcal{O}_X)$. The existence of the above resolution is a consequence of the Auslander-Gorenstein property of A_{con} . So we have

$$\mathbf{L}^{-q}j^*j_*S = \mathrm{Tor}_q^A(j_*S, A_{con}) = \begin{cases} S & q = 0, 3 \\ 0 & q = 1, 2 \end{cases}.$$

Therefore, the E_2 page has nonvanishing terms exactly for $q = 0, 3$ where both rows are isomorphic to $\mathrm{Ext}_{A_{con}}^\bullet(S, S)$. The E_4 page of the spectral sequence

has a nontrivial differential $d : \text{Ext}_{A_{con}}^p(S, S) \rightarrow \text{Ext}_{A_{con}}^{p+4}(S, S)$ for all $p \geq 0$. Because $\text{Ext}_A^n(j_*S, j_*S)$ vanish for $n \geq 4$, the differential d are isomorphisms for $p > 0$. For $p = 0$, notice that both $\text{Ext}_A^3(j_*S, j_*S)$ and $\text{Ext}_{A_{con}}^3(S, S)$ are one dimensional. Therefore, d is an isomorphism for $p = 0$.

In other words, if we denote $\text{Ext}_{A_{con}}^\bullet(S, S)$ by $A_{con}^!$ and introduce a formal variable h of degree 3 then we may construct an A_∞ -algebra $A_{con}^! \oplus A_{con}^!h$ with m_2, m_3, \dots of $A_{con}^!$ extended linearly and m_1 defined by

$$m_1(x + y \cdot h) = d(y).$$

The A_∞ -algebra $\text{Ext}_A^\bullet(j_*S, j_*S)$ is quasi-isomorphic to $A_{con}^! \oplus A_{con}^!h$ with the A_∞ -structure defined above. So m_k on $\text{Ext}_A^\bullet(j_*S, j_*S)$ are determined by m_k on $\text{Ext}_{A_{con}}^\bullet(S, S)$. This proves that \mathcal{C}^b can be reconstructed from A_{con} . \square

The above proof admits a geometric interpretation.

Remark 3.8. The Maurer-Cartan loci of the two A_∞ algebras $\text{Ext}_{A_{con}}^\bullet(S, S)$ are $\text{Ext}_A^\bullet(j_*S, j_*S)$ are isomorphic. They have isomorphic obstruction theory up to Ext^3 . The contraction algebra A_{con} is an example of singular Calabi-Yau algebra assuming the critical point of W is degenerated. The module category of A_{con} is an abelian subcategory of \mathcal{C}^b but its derived category is not equivalent to \mathcal{C}^b . Because \mathcal{C}^b is saturated while $\text{D}^b(\text{mod-}A_{con})$ is not.

4 Contraction algebra under deformation

Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g} & \mathcal{Y} \\ & \searrow & \downarrow \\ & & \mathcal{T} \end{array}$$

be a flat deformation of the flopping contraction $f : X \rightarrow Y$, where T is a Zariski open neighborhood of $0 \in \mathbb{A}^1$ such that $g_0 = f$ and $g_t : \mathcal{X}_t \rightarrow \mathcal{Y}_t$ for $t \neq 0$ is a flopping contraction. Denote the exceptional fiber of g_t by C_t and $C_0 = C$ for $t = 0$. By choosing a relative ample line bundle \mathcal{L} on \mathcal{X} , the exact sequence (2.2) defines a vector bundle \mathcal{N} such that $\mathcal{O}_{\mathcal{X}} \oplus \mathcal{N}$ is a tilting object. For each $t \in \mathcal{T}$, $N_t := (g_t)_*\mathcal{N}_t$ is a compact generator of $\text{D}_{sg}(\mathcal{Y}_t)$. As an analogy of Theorem 2.5, we define $A_{con,t}$ to be

$$A_{con,t} := \text{Hom}_{\text{D}_{sg}(\mathcal{Y}_t)}^0(N_t, N_t).$$

By theorem 3.3, the dimension of $A_{con,t}$ is equal to the Euler characteristic $\chi(N_t^{st}, N_t^{st})$.

Let l be the length of the scheme theoretical exceptional fiber C of f . The length l takes values $1, 2, \dots, 6$ with $l > 1$ if and only if C is a $(1, -3)$ curve. According to the classification theorem of Katz and Morrison [15], Y must have compound du Val singularities of types cA_1, cD_4, cE_6, cE_7 or cE_8 .

The most generic deformation of $f : X \rightarrow Y$ has the property that for $t \neq 0$, \mathcal{Y}_t has finitely many cA_1 singularity and \mathcal{X}_t contains finitely many $(-1, -1)$ curves with possibly different homology classes contracting to those singularities. More specifically, if $C_0 = C$ has length l then the exceptional curves in \mathcal{X}_t can have homology classes $j[C_0]$ for $j = 1, 2, \dots, l$. The number of the connected components of the exceptional fiber with class $j[C_0]$ is denoted by n_j . We will show that the dimension of $A_{con,t}$ stays constant under such a deformation.

The morphism g is a flopping contraction, and admits a flop

$$\begin{array}{ccc} \mathcal{X} & \overset{\text{-----}}{\longrightarrow} & \mathcal{X}^+ \\ & \searrow g & \swarrow g^+ \\ & \mathcal{Y} & \end{array}$$

such that we have the derived equivalence

$$\Phi_{\mathcal{X} \rightarrow \mathcal{X}^+}^{\mathcal{O}_{\mathcal{X} \times \mathcal{Y}, \mathcal{X}^+}} : D^b(\mathcal{X}) \simeq D^b(\mathcal{X}^+).$$

By composing the above equivalence twice, we obtain the autoequivalence

$$\Phi_{\mathcal{X}^+ \rightarrow \mathcal{X}}^{\mathcal{O}_{\mathcal{X} \times \mathcal{Y}, \mathcal{X}^+}} \circ \Phi_{\mathcal{X} \rightarrow \mathcal{X}^+}^{\mathcal{O}_{\mathcal{X} \times \mathcal{Y}, \mathcal{X}^+}} : D^b(\mathcal{X}) \simeq D^b(\mathcal{X}). \quad (4.1)$$

Let Ψ be an inverse of the equivalence (4.1), and

$$\mathcal{P} \in D^b(\mathcal{X} \times_{\mathcal{T}} \mathcal{X})$$

be the kernel object of Ψ . In [21], the second author proved the flatness of $\mathcal{H}^1(\mathcal{P})$ over \mathcal{T} , which leads to the following formula for the dimension of the the contraction algebra.

Theorem 4.1. ([21, Theorem 1.1]) *We have the following formula*

$$\dim_{\mathbb{C}} A_{con,0} = \sum_{j=1}^l j^2 \cdot n_j,$$

where l is the scheme theoretical length of C .

The next result shows that the dimension of $A_{con,t}$ stays constant under a generic deformation.

Theorem 4.2. *Let $g : \mathcal{X} \rightarrow \mathcal{Y}$ be a generic deformation of the flopping contraction mentioned above. Choose a relative ample line bundle \mathcal{L} on \mathcal{X} such that $\deg(\mathcal{L}|_{C_0}) = 1$. Let \mathcal{N} be the vector bundle on \mathcal{X} defined by the exact sequence 2.2. Define the contraction algebra $A_{con,t}$ to be $\text{Hom}_{D_{sg}^0(\mathcal{Y}_t)}(N_t, N_t)$. For $t \neq 0$, the contraction algebra $A_{con,t}$ is semi-simple and of the form*

$$A_{con,t} \cong \prod_{j=1}^l \prod_{k=1}^{n_j} \text{Mat}_j(\mathbb{C})$$

where $\text{Mat}_j(\mathbb{C})$ is the algebra of $j \times j$ complex matrices.

Proof. By the theorem of Van den Bergh [22], $\mathcal{O}_{\mathcal{X}} \oplus \mathcal{N}$ is a tilting bundle. Because \mathcal{N} is locally free, by base change $N_t := (g_*\mathcal{N})_t$ is isomorphic to $(g_t)_*\mathcal{N}_t$ and $\mathcal{O}_{\mathcal{X}_t} \oplus \mathcal{N}_t$ is a tilting object for every t .

For $t \neq 0$, we denote the connected components of the exceptional fiber of $\mathcal{X}_t \rightarrow \mathcal{Y}_t$ by $C_t^{j,k}$ where $j = 1, 2, \dots, l$ and $k = 1, 2, \dots, n_j$ for fixed j . The curve $C_t^{j,k}$ has homology class $j[C_0]$. Denote the formal completion of \mathcal{X}_t along $C_t^{j,k}$ by $\hat{X}_t^{j,k}$ and denote the restriction of \mathcal{L} to $\hat{X}_t^{j,k}$ by $L_t^{j,k}$. Then $\deg(L_t^{j,k}) = j$ for all $k = 1, \dots, n_j$.

Fix j and k , the line bundle $L_t^{j,k}$ is a j -fold tensor product of a line bundle L on $\hat{X}_t^{j,k}$ such that $\deg(L|_{C_t^{j,k}}) = 1$. In other words, we may write $L_t^{j,k} = L^j := L^{\otimes j}$. We need to compute the tilting bundle corresponding to L^j . Since we are in the $(-1, -1)$ situation, $H^1(\hat{X}_t^{j,k}, L^{-j})$ is generated by $H^1(C_t^{j,k}, \mathcal{O}(-j)) \cong \mathbb{C}^{j-1}$. We may choose its basis to be a minimal generating set of $H^1(\hat{X}_t^{j,k}, L^{-j})$. The short exact sequence 2.2 defines the vector bundle $\mathcal{M}_t^{j,k}$ on $\hat{X}_t^{j,k}$:

$$0 \longrightarrow L^{-j} \longrightarrow \mathcal{M}_t^{j,k} \longrightarrow \mathcal{O}_{\hat{X}_t^{j,k}}^{\oplus j-1} \longrightarrow 0$$

By the minimality, $\mathcal{M}_t^{j,k}$ is isomorphic to $(L^{-1})^{\oplus j}$.

Suppose that m is the cardinality of the minimal generating set of $H^1(\mathcal{X}, \mathcal{L}^{-1})$. By the argument above, the restriction of \mathcal{N} to $\hat{X}_t^{j,k}$ is a direct sum $\mathcal{M}_t^{j,k} \oplus \mathcal{O}_{\hat{X}_t^{j,k}}^{\oplus m+1-j}$. The existence of the second factor is due to the fact that the restriction of the minimal generating set of $H^1(\mathcal{X}, \mathcal{L}^{-1})$ is not minimal as a generating set of $H^1(\hat{X}_t^{j,k}, L^{-j})$.

Denote $\hat{Y}_t^{j,k}$ for the formal completion of \mathcal{Y}_t at $g_t(C_t^{j,k})$ and denote $M_t^{j,k}$ for $(g_t)_*\mathcal{M}_t^{j,k}$. The contraction algebra $A_{con,t}$ for $t \neq 0$ can be computed as

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}_{sg}(\mathcal{Y}_t)}^0(N_t, N_t) &\cong \prod_{j=1}^l \prod_{k=1}^{n_j} \mathrm{Hom}_{\mathrm{D}_{sg}(\hat{Y}_t^{j,k})}^0(M_t^{j,k} \oplus \mathcal{O}_{\hat{Y}_t^{j,k}}^{\oplus m+1-j}, M_t^{j,k} \oplus \mathcal{O}_{\hat{Y}_t^{j,k}}^{\oplus m+1-j}) \\ &= \prod_{j=1}^l \prod_{k=1}^{n_j} \mathrm{Hom}_{\mathrm{D}_{sg}(\hat{Y}_t^{j,k})}^0(M_t^{j,k}, M_t^{j,k}) \\ &= \prod_{j=1}^l \prod_{k=1}^{n_j} \mathrm{Mat}_j(\mathbb{C}). \end{aligned}$$

□

The above theorem together with the dimension formula in theorem 4.1 implies that $A_{con} := \mathrm{Hom}_{\mathrm{D}_{sg}(\mathcal{Y})}^0(N, N)$ is flat for a generic deformation \mathcal{T} . We expect the flatness of the contraction algebra to hold in full generality.

Conjecture 4.3. *Let $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{T}$ be an arbitrary flat deformation of flopping contraction. Denote N for $g_*\mathcal{N}$. The contraction algebra A_{con} is flat over \mathcal{T} . And the 0-th Hochschild (co)homology of the contraction algebra A_{con} is flat over \mathcal{T} .*

By theorem 4.2, the dimension of $\mathrm{HH}_0(A_{con,t})$ is

$$\sum_{j=1}^l n_j.$$

for $t \neq 0$. In all the known examples of $A_{con,0}$, we have verified that $\mathrm{HH}_0(A_{con,0})$ has dimension $\sum_{j=1}^l n_j$. The conjecture above suggests the deformation invariance of the dimensions of the contraction algebra and its 0-th Hochschild (co)homology. Indeed, the positive integers n_j have already appeared in the work of Bryan, Katz and Leung [4]. They coincide with the genus zero Gopakumar-Vafa (GV) invariants defined by Katz [12].

We recall genus zero GV invariants associated to a 3-dimensional flopping contraction $f: X \rightarrow Y$ with X smooth, C a smooth \mathbb{P}^1 . For $j \in \mathbb{Z}_{\geq 1}$, let M_j be the moduli space of one dimensional stable sheaves F on X satisfying $[F] = j[C]$ and $\chi(F) = 1$. There is a symmetric perfect obstruction theory on M_j , and we define

$$n_j = \int_{[M_j]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Note that M_j is topologically one point. There exist several other interpretations of n_j .

1. n_j coincides with the value of the Behrend function at the point of M_j .
2. n_j coincides with the dimension of the algebra \mathcal{O}_{M_j} .
3. Let $p \in S \subset Y$ be a general hypersurface, and $\overline{S} \subset X$ its proper transform. Let $I \subset \mathcal{O}_{\overline{S}}$ be the ideal sheaf of C in \overline{S} , and $C^{(j)} \subset \overline{S}$ be the subscheme defined by the syzygetic power $I^{(j)}$. Then n_j coincides with $\dim_{\mathbb{C}} \mathcal{O}_{\mathrm{Hilb}(X), C^{(j)}}$. This is shown in [12].
4. n_j coincides with the number of $(-1, -1)$ -curves with curve class $j[C]$ on a deformation of $X \rightarrow Y$. This is shown in [4, Section 2].

On the other hand for $j \in \mathbb{Z}_{\geq 1}$, let $\mathrm{Hilb}_j(A_{con})$ be the moduli space of surjections $A_{con} \twoheadrightarrow M$ in $\mathrm{mod}\text{-}A_{con}$ with $\dim_{\mathbb{C}} M = j$. We define

$$\mathrm{DT}_j(A_{con}) = \int_{\mathrm{Hilb}_j(A_{con})} \nu \cdot d\chi.$$

Here ν is the Behrend function on $\mathrm{Hilb}_j(A_{con})$.

Theorem 4.4. *We have the following formula*

$$1 + \sum_{j>0} \mathrm{DT}_j(A_{con}) t^j = \prod_{j \geq 1} (1 - (-1)^j t^j)^{j n_j}. \quad (4.2)$$

Proof. Let $\mathcal{C}^b \subset D^b(\text{coh}(X))$ be the subcategory consisting of $E \in D^b(\text{coh}(X))$ with $\mathbf{R}f_*E = 0$. The Van-den-Bergh's equivalence restricts to the equivalence

$$\mathcal{C}_0 = \mathcal{C}^b \cap \text{coh}(X) \xrightarrow{\sim} \text{mod-}A_{\text{con}}.$$

Note that \mathcal{C}_0 is the extension closure of $\mathcal{O}_C(-1)$. In particular, any object in \mathcal{C}_0 is a semistable sheaf with the same slope.

Now we fix a divisor $H \subset X$ which intersects with C at one point transversally. We recall the notion of parabolic stable pairs introduced in [19], applied in this situation. By definition, a H -parabolic stable pair consists of a pair (F, s) , where $F \in \mathcal{C}_0$ and $s \in F \otimes \mathcal{O}_H$, satisfying the following: for any surjection $\pi: F \twoheadrightarrow F'$ in \mathcal{C}_0 with $F' \neq 0$, we have $(\pi \otimes \mathcal{O}_H)(s) \neq 0$. Let M_j^{par} be the moduli space of parabolic stable pairs (F, s) with $[F] = j[C]$, which exists as a projective scheme by [19]. We show that there is an isomorphism of schemes

$$M_j^{\text{par}} \xrightarrow{\cong} \text{Hilb}_j(A_{\text{con}}). \quad (4.3)$$

Indeed, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{\Phi} & \text{mod-}A_{\text{con}} \\ & \searrow \otimes \mathcal{O}_H & \swarrow \text{forg} \\ & \mathbb{C}\text{-Vect} & \end{array}$$

Hence giving $s \in F \otimes \mathcal{O}_H$ for $F \in \mathcal{C}_0$ is equivalent to giving an element of $\tau \in \Phi(F)$. Moreover (F, s) is parabolic stable if and only if $\tau \in \Phi(F)$ generates $\Phi(F)$ as a right A_{con} -module. Therefore we obtain the isomorphism (4.3).

We define $\text{DT}_j^{\text{par}} \in \mathbb{Z}$ to be

$$\text{DT}_j^{\text{par}} = \int_{M_j^{\text{par}}} \nu' \cdot d\chi.$$

Here ν' is the Behrend function on M_j^{par} . By [20, Proposition 3.16], we have the formula

$$1 + \sum_{j>0} \text{DT}_j^{\text{par}} t^j = \prod_{j \geq 1} (1 - (-1)^j t^j)^{jn_j}.$$

By the isomorphism (4.3), we have $\text{DT}_j^{\text{par}} = \text{DT}_j(A_{\text{con}})$, hence obtain the desired result. \square

As a corollary, we reconstruct the result of [21].

Corollary 4.5. *(Theorem 4.1) We have the formula*

$$\dim_{\mathbb{C}} A_{\text{con}} = \sum_{j=1}^l j^2 n_j.$$

Proof. Since $\text{Hilb}_j(A_{\text{con}}) = \emptyset$ for $j > \dim_{\mathbb{C}} A_{\text{con}}$ and coincides with $\text{Spec } \mathbb{C}$ for $j = \dim_{\mathbb{C}} A_{\text{con}}$, the equation (4.2) is written as

$$t^{\dim_{\mathbb{C}} A_{\text{con}}} + O(t^{\dim_{\mathbb{C}} A_{\text{con}}-1}) = t^{\sum_{j=1}^l j^2 n_j} + O(t^{\sum_{j=1}^l j^2 n_j - 1}).$$

□

We can also characterize n_j and l from the contraction algebra.

Corollary 4.6. *Let $f': X' \rightarrow Y'$ be another 3-fold flopping contraction with X' smooth and the exceptional curve C' being a smooth \mathbb{P}^1 . Let A'_{con} , l' and n'_j be the corresponding contraction algebra, length and the GV invariants. If $A_{\text{con}} \cong A'_{\text{con}}$ as algebras, we have $n_j = n'_j$ for all $j \geq 1$. In particular, we have $l = l'$.*

Proof. The formula (4.2) gives

$$\prod_{j \geq 1} (1 - (-1)^j t^j)^{j n_j} = \prod_{j \geq 1} (1 - (-1)^j t^j)^{j n'_j}.$$

By comparing the coefficients of the above equality, we easily obtain $n_j = n'_j$ for all $j \geq 1$. Since we have

$$l = \max\{j \geq 1 : n_j \neq 0\},$$

we in particular obtain $l = l'$. □

5 The ∞ -contraction algebra

In this section, we study the A_∞ -structure on A_{con}^∞ . When the singularity is weighted homogeneous, the Milnor algebra completely determine the local ring of the singularity. In the general cases, the Milnor algebra is not sufficient to determine the local ring. The additional structure needed is the action of the defining equation of the hypersurface on the Milnor algebra.

Let A and B be two \mathbb{Z} -graded or $\mathbb{Z}/2$ -graded A_∞ algebras. We say B is *derived Morita equivalent* with A if there exists an A_∞ $A - B$ -bimodule X such that the functor $- \otimes_A^L X$ induces a derived equivalence

$$DA \cong DB$$

where DA and DB are the derived categories of modules over A and B . The following proposition shows that the ∞ -contraction algebra recovers the Milnor algebra.

Proposition 5.1. *Suppose that $\hat{X} \rightarrow \hat{Y}$ and $\hat{X}' \rightarrow \hat{Y}'$ are three dimensional flopping contractions such that \hat{Y} and \hat{Y}' are the spectrum of $\mathbb{C}[[x, y, z, w]]/(W)$ and $\mathbb{C}[[x, y, z, w]]/(W')$ respectively, for germs of functions W and W' . The Milnor algebras B_W and $B_{W'}$ are isomorphic as algebras if the corresponding ∞ -contraction algebras are derived Morita equivalent.*

Proof. By the derived Morita equivalence of $\mathbb{Z}/2$ -graded dg-category and proposition 3.1, $D_{sg}(\hat{Y})$ is equivalent with the triangulated category of the dg category of modules over A_{con}^∞ . Therefore, the Hochschild cohomology of $MF(W) \cong D_{sg}(\hat{Y})$ is isomorphic to the Hochschild cohomology of A_{con}^∞ as algebras. By corollary 6.4 of [6], $HH^\bullet(MF(W))$ is isomorphic to the Milnor algebra B_W as algebras. \square

In [7], Donovan and Weymss conjecture that the local ring of Y at the singularity p is determined by the contraction algebra.

Conjecture 5.2. ([7, Conjecture 1.4]) *Suppose that $X \rightarrow Y$ and $X' \rightarrow Y'$ are three dimensional flopping contractions in smooth quasi-projective 3-folds, to points p and p' respectively. To these, associate the contraction algebras A_{con} and B_{con} . Then the completions of stalks at p and p' are isomorphic if and only if $A_{con} \cong B_{con}$.*

From theorem 4.2, we see that the different choices of the ample line bundles may lead to Morita equivalent contraction algebras. So it is natural to replace the condition $A_{con} \cong B_{con}$ in the above conjecture by the condition that A_{con} and B_{con} being Morita equivalent. However, this may not be sufficient. In section 6, we will see that for almost all examples the A_∞ -structure on the ∞ -contraction algebra is nontrivial.

We propose the following variation of the conjecture of Donovan and Wemyss and prove one special case.

Conjecture 5.3. *Suppose that $X \rightarrow Y$ and $X' \rightarrow Y'$ are three dimensional flopping contractions in smooth quasi-projective 3-folds, to points p and p' respectively. To these, associate the ∞ -contraction algebras A_{con}^∞ and B_{con}^∞ . Then the completions of stalks at p and q are isomorphic if and only if A_{con}^∞ and B_{con}^∞ are derived Morita equivalent.*

Let f and g be two germs of holomorphic (or formal) functions on $(\mathbb{C}^n, 0)$. We say f and g are *right equivalent* if they differ by a holomorphic (or formal) change of variables on the domain. We say f is *weighted homogeneous* if f is right equivalent with a weighted homogeneous polynomial. In section 6, we give several examples of compound du Val singularities that admits small resolutions. Among these examples, the type cA_1 singularities and the Laufer's cD_4 singularity are weighted homogeneous.

The following result is a consequence of the famous theorem of Mather and Yau.

Theorem 5.4. (Proposition 2.5, Theorem 4.2 [5]) *Let $f, g : (\mathbb{C}^n, 0) \rightarrow \mathbb{C}$ be two germs of holomorphic functions with isolated singularities at the origin. Denote the corresponding completions of the rings of functions on the hypersurfaces at the origin by $\widehat{\mathcal{O}}/(f)$ and $\widehat{\mathcal{O}}/(g)$ and denote the corresponding Milnor algebras by B_f and B_g . Assume that f and g are weighted homogeneous. Then $\widehat{\mathcal{O}}/(f)$ is isomorphic to $\widehat{\mathcal{O}}/(g)$ if and only if $B_f \cong B_g$ as algebras.*

Theorem 5.5. *Assume that Y and Y' have weighted homogeneous hypersurface singularities. Conjecture 5.3 holds.*

Proof. By theorem 5.4, the local rings of the weighted homogeneous isolated hypersurface singularities are isomorphic if and only if their Milnor algebras are isomorphic. Then by proposition 5.1, the Milnor algebra can be reconstructed from the infinity contraction algebra by taking its Hochschild cohomology. So the conjecture is proved. \square

To finish this section, we sketch a possible way to prove conjecture 5.3 for the general cases. And we will return to this in the future work.

The Milnor algebra B_f is equipped with a $\mathbb{C}[t]$ -algebra structure by letting t acts by multiplying with f , treated as an element in B_f . This action is trivial if f is weighted homogeneous since f belongs to the jacobian ideal J_W . However, it is not the case in general (see the cE_6 singularity example in section 6). Though the \mathbb{C} -algebra structure on B_f does not determine the local ring of the singularity in general (see the counter example in [5]), the $\mathbb{C}[t]$ -algebra structure on B_f determines $\widehat{\mathcal{O}}/(f)$ (cf. [10, Theorem 2.28]). The conjecture 5.3 can be reduced to the problem of finding a categorical construction for the action of f on B_f .

Let A be a \mathbb{Z} -graded or $\mathbb{Z}/2$ -graded A_∞ -algebra. Then Hochschild cochain complex

$$C^\bullet(A, A) := \text{Hom}^\bullet\left(\bigoplus_{n \geq 0} A[1]^{\otimes n}, A\right)$$

is a B_∞ -algebra in the sense of Getzler and Jones (section 5.2 [9]). The B_∞ -structure induces the cup product and Gerstenhaber bracket on the Hochschild cohomology. Keller proved that the derived Morita equivalences preserve the homotopy type of $C^\bullet(A, A)$ as B_∞ -algebras.

Theorem 5.6. ([14, Section 3.2]) *Let A and B be dg-algebras and let X be a dg $A - B$ -bimodule. Suppose the functor*

$$- \otimes_A^L X : \text{DA} \rightarrow \text{DB}$$

is an equivalence, there is a canonical isomorphism

$$\phi_X : C^\bullet(B, B) \rightarrow C^\bullet(A, A)$$

in the homotopy category of B_∞ -algebras.

Take B to be the ∞ -contraction algebra A_{con}^∞ . We speculate that the W can be lifted to an even cocycle in $C^\bullet(B, B)$ using the B_∞ -structure such that its cohomology class is invariant under the derived Morita equivalence.

6 Examples

In this section, we study the (∞ -)contraction algebras associated to three types of singularities. The first class of examples are compound du Val singularities

of type cA_1 that admit small resolution by a length 1 curve. They are classified by a positive integer k called width. The normal bundle of the exceptional curve is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ if $k = 1$ and $\mathcal{O} \oplus \mathcal{O}(-2)$ when $k > 1$. In the $k = 1$ case, the contraction algebra has trivial A_∞ -structure. However, for $k > 1$ the contraction algebra carries nontrivial A_∞ structure.

The second example have singularities of type cD_4 . Its scheme theoretical fiber has length 2. We calculate the dimension of the contraction algebra using the Riemann-Roch formula of the matrix factorization. This calculation requires no knowledge about the ring structure on A_{con} . Moreover, we compute the boundary bulk map and the Frobenius structure explicitly and verify conjecture 4.3.

In [23], Weymss and Brown compute the contraction algebra of a cE_6 flopping contraction. In this example, the fiber has length 3. This is a first non-homogeneous example. Based on the result of [23], we compute the boundary bulk map for it. It turns out that conjecture 4.3 also holds.

6.1 Length 1 case

Consider the hypersurface singularity

$$W = x^2 - y^{2k} + zw = 0,$$

where k is a positive integer. Denote R for $\mathbb{C}[[x, y, z, w]]/(W)$ and Y for $\text{Spec } R$. It admits a 2×2 matrix factorization

$$\Psi = \begin{pmatrix} w & -x - y^k \\ x - y^k & z \end{pmatrix}, \quad \Phi = \begin{pmatrix} z & x + y^k \\ -x + y^k & w \end{pmatrix}.$$

Set $N = \text{coker}(\Psi)$. As usual, we denote the above matrix factorization by N^{st} . Y admits two small resolutions X and X^+ such that the exceptional fibers are both a smooth rational curve of length 1. Denote the exceptional curve of $f : X \rightarrow Y$ by C . The normal bundle of C is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ if $k = 1$ and $\mathcal{O} \oplus \mathcal{O}(-2)$ if $k > 1$. The number k is referred as the *width*. By the theorem of Van den Bergh [22], $R \oplus N$ is a tilting object for $D^b(\text{coh}(X))$ (cf. [1, Section 3.3]). The scalar multiplication by y defines an endomorphism of N^{st} , and therefore an element in A_{con} . The dimension of A_{con} is computed by the Riemann-Roch formula of matrix factorization (3.2). We obtain $\chi(N^{st}, N^{st}) = k$. One can verify that A_{con} is isomorphic to $\mathbb{C}[a]/(a^k)$ with a acts by scalar multiplication with y . The Milnor algebra B_W is $\mathbb{C}[y]/(y^{2k-1})$. The boundary bulk map τ sends a^n to y^{k-1+n} for $n = 0, 1, \dots, k-1$. The socle ideal (a^{k-1}) is mapped to the socle ideal (y^{2k-1}) . The kernel of the Frobenius trace on A_{con} is $\mathbb{C}\{1, a, \dots, a^{k-2}\}$.

Consider the case $k = 2$. The contraction algebra carries nontrivial A_∞ structure. By the calculation above, A_{con} is isomorphic to the algebra of dual number $\mathbb{C}[\epsilon]/\epsilon^2$ and the B_W is $\mathbb{C}[y]/y^3$. The bar complex $\text{Bar}(\mathbb{C}[\epsilon]/\epsilon^2)$ is

$$0 \longleftarrow \mathbb{C} \xleftarrow{0} \mathbb{C}u^{-1} \xleftarrow{0} \mathbb{C}u^{-2} \longleftarrow \dots$$

with $\deg(u) = 1$. The normalized Hochschild cochain $C^n = \text{Hom}(\mathbb{C}u^n, \mathbb{C}[\epsilon]/\epsilon^2)$ can be identified with $\mathbb{C}[\epsilon]/\epsilon^2$ by $u^n \mapsto a + b\epsilon$. The differential Q acts by

$$\begin{aligned} Q_0 &= 0 \\ Q_n(a + b\epsilon) &= (1 + (-1)^{n+1})a\epsilon, \quad n > 1. \end{aligned}$$

As a consequence,

$$\text{HH}^i(\mathbb{C}[\epsilon]/\epsilon^2) = \begin{cases} \mathbb{C}[\epsilon]/\epsilon^2 & i = 0; \\ \mathbb{C} & i = 2k, k > 0; \\ \mathbb{C} & i = 2k + 1, k \geq 0 \end{cases}$$

On the other hand, because N^{st} is a compact generator of $\text{MF}(W)$ the Hochschild cohomology of A_{con}^∞ is isomorphic to the Milnor algebra $B_W = \mathbb{C}[w]/w^3$. Using the method developed in [1], we compute the A_∞ structure on A_{con}^∞ . It turns out that there is a higher Massey product

$$m_4(\epsilon, \epsilon, \epsilon, \epsilon) = 1.$$

In this example, $n_1 = 2$ and $n_j = 0$ for $j > 0$. The dimension of the Milnor algebra is 3. Under a generic deformation, $W = 0$ is deformed to two cA_1 singularities of $k = 1$. So we see that the dimension of $\text{HH}_0(A_{con})$ is a deformation invariant while the dimension of the Milnor algebra is not.

6.2 Laufer's flop

The following example, called Laufer's flop, is studied in [1, Section 4.4] and [7].

Consider the hypersurface singularity

$$W = x^2 + y^3 + wz^2 + w^{2k+1}y = 0$$

for $k > 0$. Notice that it is a weighted homogeneous hypersurface singularity with weight

$$\left(\frac{6k+3}{4}, \frac{2k+1}{2}, \frac{6k+1}{4}, 1\right).$$

It admits a matrix factorization:

$$\Psi = \begin{pmatrix} x & y & z & w^k \\ -y^2 & x & -yw^k & z \\ -wz & w^{k+1} & x & -y \\ -yw^{k+1} & -wz & y^2 & x \end{pmatrix}, \quad \Phi = \begin{pmatrix} x & -y & -z & -w^k \\ y^2 & x & yw^k & -z \\ wz & -w^{k+1} & x & y \\ yw^{k+1} & wz & -y^2 & x \end{pmatrix}.$$

Denote $\text{coker}(\Psi)$ by N . One can check that $R \oplus N$ is a tilting object. We calculate that $ch(N^{st}) = 2(6k+3)yw^k$. To compute the residue, we need to make a change of variable

$$\begin{pmatrix} 2x \\ 3y^3 \\ \frac{2z^3}{2k+1} \\ \frac{(2k+1)w^{4k+2}}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y & \frac{z}{2(2k+1)} & \frac{-w}{2k+1} \\ 0 & 0 & -w^{2k-1}y & \frac{2z}{2k+1} \\ 0 & \frac{(2k+1)w^{2k+1}}{3} & \frac{yz}{2} & -yw \end{pmatrix} \cdot \begin{pmatrix} \partial_x W \\ \partial_y W \\ \partial_z W \\ \partial_w W \end{pmatrix}$$

The determinant of the above matrix is

$$\begin{aligned}
& y(y^2 w^{2k} - \frac{yz^2}{2k+1}) + \frac{2k+1}{3} w^{2k+1} (\frac{z^2}{(2k+1)^2} - \frac{w^{2k}y}{2k+1}). \\
\chi(N^{st}, N^{st}) &= \text{Res} \left[\frac{4(6k+3)^2 y^2 w^{2k} \cdot dx_1 \wedge \dots \wedge dx_n}{\partial_1 W, \dots, \partial_n W} \right] \\
&= \text{Res} \frac{4(6k+3)^2 y^2 w^{2k} \cdot \frac{w^{2k+1} z^2}{6k+3}}{2x \cdot 3y^2 \cdot \frac{2z^2}{2k+1} \cdot \frac{2k+1}{3} w^{4k+2}} \\
&= (6k+3) \text{Res} \frac{1}{xyzw} \\
&= 6k+3
\end{aligned}$$

So A_{con} has dimension $6k+3$. Using the method in [1], Donovan and Wemyss computed the algebra structure and obtain

$$A_{con} \cong \mathbb{C}\langle a, b \rangle / (ab + ba, a^2 - b^{2k+1}).$$

(See [7, Example 3.14].)

We focus on the simplest case of $k=1$. The generators a and b are degree zero morphisms of N^{st} . They are represented by the matrices

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -y & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \end{pmatrix}$$

(cf. [1, Section 4.4].)

The Milnor algebra B_W is a vector space of dimension 11 with basis

$$1, y, y^2, y^2 w, yz, yz^2, yw, z, z^2, w, w^2.$$

The socle ideal is spanned by the element $yz^2 = -3y^2 w^2$. We compute the boundary bulk map from A_{con} to B_W

$$\begin{aligned}
\tau(1) &= -18yw, & \tau(a) &= 6yz, & \tau(b) &= 6w^3 \\
\tau(a^2) &= 18y^2 w, & \tau(a^2 b) &= -6yw^3 = 0, & \tau(b^2) &= 18yw^2, & \tau(a^2 b^2) &= 18y^2 w^2.
\end{aligned}$$

The commutator $[A_{con}, A_{con}]$ consists of elements $a^2 b, ab$ and ab^2 . By proposition 3.6, the boundary bulk morphism will map them to zero. The socle ideal ($a^2 b^2$) of A_{con} maps to the socle ideal of B_W .

In this example, $n_1 = 5$, $n_2 = 1$ and $n_j = 0$ for $j > 0$. The first Hochschild homology $\text{HH}_0(A_{con})$ is $A_{con}/[A_{con}, A_{con}]$. As a vector space, it is spanned by $1, a, a^2, b, b^2, a^2 b^2$. So the dimension is $n_1 + n_2$.

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